

CMV MATRICES AND LITTLE AND BIG -1 JACOBI POLYNOMIALS

MAXIM DEREVYAGIN, LUC VINET, AND ALEXEI ZHEDANOV

ABSTRACT. We introduce a new map from polynomials orthogonal on the unit circle to polynomials orthogonal on the real axis. This map is closely related with the theory of CMV matrices. It contains an arbitrary parameter λ which leads to a linear operator pencil. We show that the little and big -1 Jacobi polynomials are naturally obtained under this map from the Jacobi polynomials on the unit circle.

1. INTRODUCTION

In this paper we propose a new map from polynomials orthogonal on the unit circle (OPUC) to polynomial orthogonal on the real line. This map can be considered as a one-parameter generalization of the Delsarte-Genin [19] map from OPUC to symmetric orthogonal polynomials on an interval. We derive this map from elementary properties of the CMV matrices. The CMV matrices are known to play a crucial role in the theory of OPUC [31], [35], [11].

This new map contains an additional arbitrary parameter λ which leads to infinitely many families of polynomials $P_n(x; \lambda)$ orthogonal on the real line. In general, for arbitrary real values of λ these polynomials are orthogonal on a union of two distinct intervals of the real line.

Starting from given explicit sets of OPUC, one thus constructs families of polynomials orthogonal on the real line $P_n(x; \lambda)$. The main result of this paper consists in the following. Starting from the Jacobi polynomials on the unit circle, we construct a one-parameter family of polynomials $P_n(x; \lambda)$ orthogonal on the union of two intervals of the real line.

We identify these polynomials with the big -1 Jacobi polynomials discovered recently in [38]. For a particular value $\lambda = 1$ of the parameter we obtain the little -1 Jacobi polynomials found in [37].

The underlying spectral problem has the following form

$$(1.1) \quad (L + \lambda M - xI)\vec{q}(x, z) = 0,$$

where L and M are tridiagonal matrices of a special form (L and M have zeros on the super- and sub-diagonals).

If we fix λ , the problem (1.1) is a classical eigenvalue problem for the tridiagonal matrix $K(\lambda) = L + \lambda M$. Thus, it gives rise to a system of orthogonal polynomials, Padé approximants and moment problems.

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If we fix x , the problem (1.1) is a generalized eigenvalue problem for the two matrices M and $L - xI$. It generates a system of functions rational in x that are related to the theory of bi-orthogonal rational functions [43]. Moreover, it has recently been shown that such generalized eigenvalue problems appear in rational interpolation [6], [18], Nevanlinna-Pick problems [17], [18], integrable systems [33], [36], and mechanics [24, Section 2.4] (see also [20, 21]).

A priori, the general theory associated to the problem (1.1) is rather involved. That is why we make some assumptions on the matrices L and M . Namely, we assume that $L^2 = I$ and $M^2 = I$. Thus, the main idea is to identify systems of orthogonal polynomials that have a hidden biorthogonality associated to a parameter.

Note that the special case $x = 0$ leads to the Schur linear pencil $L + \lambda M$ considered by Watkins [35].

Finally it should be mentioned that particular cases of (1.1) surprisingly arise in the theory of martingale polynomials [9], [30], two-state free Brownian motions [1], and Gaussian processes [27]. Let us also signal references [22] and [16] where similar pencils appear in the investigation of exactly solvable kinetic models and the corresponding combinatorial problems.

The paper is organized as follows.

In Section 2, we recall elementary facts concerning OPUC and CMV matrices.

In Section 3, we introduce the Jacobi (3-diagonal) matrix J starting from two unitary elementary matrices L and M . A new family $Q_n(x)$ of orthogonal polynomials on the real line is associated with the matrix J .

In Section 4, basic results on the Delsarte-Genin map from symmetric orthogonal polynomials $S_n(x)$ to OPUC are presented.

In Section 5, we introduce the companion Delsarte-Genin polynomials $\tilde{P}_n(x)$ as Christoffel transforms of the polynomials $S_n(x)$. We show that the polynomials $Q_n(x)$ coincide with $\tilde{P}_n(x)$.

In Section 6, we consider the Jacobi OPUC and show that the corresponding polynomials $Q_n(x)$ coincide with the little -1 Jacobi polynomials.

In Section 7, a one parameter generalization of the previous scheme - namely the Schur-Delsarte-Genin (SDG) map - is proposed. Starting from the same unitary matrices L and M we construct a 3-diagonal matrix $K(\lambda) = L + \lambda M$ that depends on a parameter λ . The matrix J corresponds to $\lambda = 1$. The matrix $K(\lambda)$ generates a new family of orthogonal polynomials $Q_n(x; \lambda)$ depending on the parameter λ . A linear operator pencil is associated with the eigenvalue problem for the matrix $K(\lambda)$.

In Section 8, we consider the special case $a_0 = a_1 = \dots = 0$ that corresponds to the OPUC $\Phi_n(z) = z^n$. The polynomials $Q_n(x; \lambda)$ are then shown to be orthogonal on the union of two intervals of the real axis.

In order to recover the orthogonality measure for the general case, we need a special map from symmetric to non-symmetric orthogonal polynomials that has been proposed by Chihara. This leads to a nontrivial linear operator pencil $J_1 + \lambda J_2$. This technique is described in Sections 8 and 9.

In Section 10 we relate the polynomials $Q_n(x; \lambda)$ to special polynomials $P_n(x)$ independent of λ and having a known orthogonality measure.

Finally in Section 11, we apply this technique to Jacobi OPUC. The corresponding polynomials $Q_n(x; \lambda)$ are shown to coincide with the big -1 Jacobi polynomials which are orthogonal on the union of two intervals on the real axis.

2. POLYNOMIALS ORTHOGONAL ON THE UNIT CIRCLE AND CMV MATRICES

Monic polynomials $\Phi_n(z) = z^n + O(z^{n-1})$ on the unit circle are defined through the recurrence relation

$$(2.1) \quad \Phi_{n+1}(z) = z\Phi_n(z) - \bar{a}_n\Phi_n^*(z),$$

with initial condition $\Phi_0(z) = 1$ where

$$\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})}.$$

The recurrence coefficients $a_n = -\overline{\Phi_{n+1}(0)}$ are called reflection parameters (sometimes also referred as the Schur, Geronimus, Verblunsky, ... parameters). It is well known that the reflection parameters are characterized by the property $|a_n| < 1$ for $n = 0, 1, 2, \dots$.

In what follows we will consider only the case where a_n are real.

For convenience, we set

$$(2.2) \quad a_{-1} = -1.$$

The dual recurrence relation is

$$(2.3) \quad \Phi_{n+1}^*(z) = \Phi_n^*(z) - a_n z \Phi_n(z).$$

It is standard to introduce the parameters

$$r_n = \sqrt{1 - a_n^2}$$

(the arithmetic meaning of the square root is assumed).

Following [35] and [11], let us introduce the block-diagonal matrix

$$(2.4) \quad L = \begin{pmatrix} a_0 & r_0 & & & & \\ r_0 & -a_0 & & & & \\ & & a_2 & r_2 & & \\ & & r_2 & -a_2 & & \\ & & & & a_4 & r_4 \\ & & & & r_4 & -a_4 \\ & & & & & \ddots \end{pmatrix}$$

and the block-diagonal matrix

$$(2.5) \quad M = \begin{pmatrix} 1 & & & & & \\ & a_1 & r_1 & & & \\ & r_1 & -a_1 & & & \\ & & & a_3 & r_3 & \\ & & & r_3 & -a_3 & \\ & & & & & a_5 & r_5 \\ & & & & & r_5 & -a_5 \\ & & & & & & \ddots \end{pmatrix}.$$

Both L and M are block-diagonal unitary matrices. Define also the 5-diagonal unitary matrix U [35], [11]

$$(2.6) \quad U = LM$$

(which is called the "Zigzag matrix" in [35] and the "CMV matrix" in [31]).

In [28] it was proposed to consider the Hermitian matrix

$$(2.7) \quad H = U + U^\dagger.$$

The authors of [28] connected this matrix to a direct sum of two Jacobi matrices and then arrived at the famous Szegő relations between polynomials orthogonal on the unit circle and on the real line. In the next section we propose a slightly different approach to the matrix H which connects the polynomials on the unit circle with a more simple class of polynomials orthogonal on the interval. These polynomials are related to symmetric orthogonal polynomials on the same interval by a simple Christoffel transform.

3. FROM CMV MATRICES TO POLYNOMIALS ORTHOGONAL ON THE REAL LINE

On the natural basis e_n , $n = 0, 1, 2, \dots$ the matrix H is 5-diagonal and Hermitian

$$(3.1) \quad \begin{aligned} H e_n = & r_n r_{n+1} e_{n+2} + r_n (a_{n+1} - a_{n-1}) e_{n+1} - 2a_n a_{n-1} e_n + \\ & r_{n-1} (a_n - a_{n-2}) e_{n-1} + r_{n-1} r_{n-2} e_{n-2}. \end{aligned}$$

Introduce the 3-diagonal matrix J which is defined by

$$(3.2) \quad J e_n = r_n e_{n+1} + (a_n - a_{n-1}) e_n + r_{n-1} e_{n-1}.$$

It is easily verified that

$$(3.3) \quad H = J^2 - 2.$$

Thus among the eigenvectors of the matrix H there are the eigenvectors of the matrix J . Moreover, it is seen that

$$(3.4) \quad J = L + M.$$

Note that the matrices L and M are both unitary and satisfy the obvious property $L^2 = M^2 = 1$. Hence

$$(3.5) \quad J^2 = (L + M)^2 = L^2 + M^2 + LM + ML = 2 + U + U^* = 2 + H,$$

thus recovering relation (3.3).

The matrix J is a symmetric Jacobi matrix. It generates orthonormal polynomials $\hat{Q}_n(x)$ satisfying the recurrence relation

$$(3.6) \quad r_{n+1} \hat{Q}_{n+1}(x) + (a_n - a_{n-1}) \hat{Q}_n(x) + r_n \hat{Q}_{n-1}(x) = x \hat{Q}_n(x).$$

The corresponding monic orthogonal polynomials

$$(3.7) \quad Q_n(x) = \frac{\hat{Q}_n(x)}{r_1 r_2 \dots r_n} = x^n + O(x^{n-1})$$

satisfy the recurrence relation

$$(3.8) \quad Q_{n+1}(x) + b_n Q_n(x) + u_n Q_{n-1}(x) = x Q_n(x)$$

with initial conditions

$$Q_0 = 1, \quad Q_1(x) = x - b_0,$$

where the recurrence coefficients are

$$(3.9) \quad u_n = r_{n-1}^2 = 1 - a_{n-1}^2, \quad b_n = a_n - a_{n-1}.$$

Note that

$$u_0 = 0, \quad b_0 = a_0 + 1$$

due to the assumption $a_{-1} = -1$.

The relation between z and x is given by

$$(3.10) \quad x = z^{1/2} + z^{-1/2}.$$

Here it is assumed that one branch of the two-valued analytic function $z^{1/2}$ is chosen. In what follows we shall take $z^{1/2} = r^{1/2}e^{i\phi/2}$ when $z = re^{i\phi}$, $0 \leq \phi \leq 2\pi$. Suppose that a point z is moving counterclockwise on the unit circle $|z| = 1$ starting from the point $z = 1$ and returning to the same point after a full rotation. Then the corresponding image point $x = z^{1/2} + z^{-1/2}$ is moving in one direction on the interval $[-2, 2]$ starting from the point 2 and ending at the point -2 . Thus the mapping (3.10) is a one-to-one correspondence between the punctured unit circle (i.e. $z \neq 1$) and the interval $[-2, 2]$. The only exception is the point $z = 1$: it corresponds to the two endpoints ± 2 of the interval.

Generic polynomials $\Phi_n(z)$ under condition $|a_n| < 1$ (with complex coefficients a_n) are orthogonal on the unit circle

$$(3.11) \quad \int_0^{2\pi} \Phi_n(e^{i\theta}) \bar{\Phi}_m(e^{-i\theta}) d\nu(\theta) = 0, \quad n \neq m,$$

where $\nu(\theta)$ is a nondecreasing function in the interval $[0, 2\pi]$.

For the case of real parameters a_n we will assume that $d\nu(\theta) = \rho(\theta)d\theta$, where $\rho(\theta)$ is a positive weight function:

$$(3.12) \quad \int_0^{2\pi} \Phi_n(e^{i\theta}) \Phi_m(e^{-i\theta}) \rho(\theta) d\theta = 0, \quad n \neq m.$$

The function $\rho(\theta)$ satisfy the symmetric property

$$(3.13) \quad \rho(2\pi - \theta) = \rho(\theta)$$

which is valid for real coefficients a_n verifying the condition $-1 < a_n < 1$ [31].

To end this section, note that $Q_n(x)$ are polynomials orthogonal with respect to a measure. In fact, this measure can be found by means of the map proposed by Delsarte and Genin.

4. THE DELSARTE-GENIN MAP TO SYMMETRIC POLYNOMIALS ON THE INTERVAL

Consider the symmetric polynomials $S_n(x)$ satisfying the recurrence relation

$$(4.1) \quad S_{n+1}(x) + v_n S_{n-1}(x) = x S_n(x),$$

where

$$(4.2) \quad v_n = (1 + a_{n-1})(1 - a_{n-2})$$

Delsarte and Genin showed [19] that the polynomials $S_n(x)$ are related to the polynomials $\Phi_n(z)$ as follows (see also [40])

$$(4.3) \quad S_n(x) = \frac{z^{-n/2}(\Phi_n(z) + \Phi_n^*(z))}{1 - a_{n-1}}.$$

The reciprocal map is

$$(4.4) \quad \begin{aligned} \Phi_n(z) &= \frac{z^{n/2}(z^{1/2}S_{n+1}(x) - \sigma_n S_n(x))}{z-1}, \\ \Phi_n^*(z) &= \frac{z^{(n+1)/2}(z^{1/2}S_{n+1}(x) - \sigma_n z^{1/2}S_n(x))}{1-z} \end{aligned}$$

where

$$\sigma_n = \frac{S_{n+1}(2)}{S_n(2)}.$$

The reflection parameters a_n are expressed as

$$(4.5) \quad a_{n-1} = 1 - \sigma_n.$$

The symmetric orthogonal polynomials $S_n(x)$ are orthogonal on the interval $[-2, 2]$:

$$(4.6) \quad \int_{-2}^2 S_n(x) S_m(x) w(x) dx = 0, \quad n \neq m,$$

where the weight function $w(x)$ is symmetric $w(-x) = w(x)$ and is related to the weight function $\rho(\theta)$ in the orthogonality relation (3.12) as follows [19, 40]

$$(4.7) \quad w(x) = \frac{\rho(\theta)}{\sin(\theta/2)}, \quad x = 2 \cos(\theta/2).$$

Note that $\sin(\theta/2) > 0$ for $0 < \theta < 2\pi$ and that the symmetry property (3.13) is equivalent to the symmetry property $w(-x) = w(x)$.

There is an obvious "scaling" generalization of the Delsarte-Genin mapping [40]. For an arbitrary positive parameter g define the map

$$(4.8) \quad S_n(x; g) = \frac{g^n z^{-n/2}(\Phi_n(z) + \Phi_n^*(z))}{1 - a_{n-1}},$$

where the relation between x and z is

$$(4.9) \quad x = g(z^{1/2} + z^{-1/2}).$$

It is easily seen that the polynomials $S_n(x; g)$ are symmetric orthogonal polynomials satisfying the recurrence relation

$$(4.10) \quad S_n(x; g) + g^2 v_n S_{n-1}(x; g) = x S_n(x)$$

with v_n also given by (4.2). The polynomials $S_n(x)$ introduced in (4.3) correspond to the case $g = 1$. From (4.10) it is clear that

$$(4.11) \quad S_n(x; g) = g^n S_n(x/g)$$

i.e. the polynomials $S_n(x; g)$ are obtained from the polynomials $S_n(x)$ by a scaling transformation of the argument x . In particular, if the polynomials $S_n(x)$ were orthogonal on the interval $[-2, 2]$, the polynomials $S_n(x; g)$ are orthogonal on the interval $[2g, -2g]$.

The reciprocal transformation is given by the formulas

$$(4.12) \quad \Phi_n(z) = g^{-n-1} \frac{z^{n/2}(z^{1/2}S_{n+1}(x; g) - \sigma_n(g)S_n(x; g))}{z-1},$$

where

$$\sigma_n(g) = \frac{S_{n+1}(2g; g)}{S_n(2g; g)} = g\sigma_n,$$

and

$$\sigma_n = \sigma_n(1) = \frac{S_{n+1}(2)}{S_n(2)}.$$

5. THE COMPANION DELSARTE-GENIN MAP AND ORTHOGONALITY MEASURES FOR SDG-TRANSFORMED POLYNOMIALS

In this section we find the orthogonality measure for $Q_n(x)$ by performing a Christoffel transformation of the polynomials from the previous section. The resulting map will be called the Schur-Delsarte-Genin map (or, shortly, the SDG-map).

First, consider the polynomials $\tilde{P}_n(x)$ which are obtained from $S_n(x)$ by the Christoffel transform

$$(5.1) \quad \tilde{P}_n(x) = \frac{S_{n+1} - A_n S_n(x)}{x + 2},$$

where

$$(5.2) \quad A_n = \frac{S_{n+1}(-2)}{S_n(-2)} = -\sigma_n = a_{n-1} - 1$$

(the last formula follows from the property $S_n(-x) = (-1)^n S_n(x)$ for the symmetric orthogonal polynomials).

The polynomials $\tilde{P}_n(x)$ satisfy the recurrence relation

$$(5.3) \quad \tilde{P}_{n+1}(x) + \tilde{b}_n \tilde{P}_n(x) + \tilde{u}_n \tilde{P}_{n-1}(x) = \tilde{P}_n(x),$$

where (see, e.g. [42]) the new recurrence coefficients are

$$(5.4) \quad \tilde{u}_n = v_n \frac{A_n}{A_{n-1}}, \quad \tilde{b}_n = A_{n+1} - A_n.$$

From (5.4) and (4.5), we immediately see that the recurrence coefficients \tilde{b}_n, \tilde{u}_n coincide with (3.9):

$$\tilde{u}_n = 1 - a_{n-1}^2, \quad \tilde{b}_n = a_n - a_{n-1}.$$

This means that the polynomials $\tilde{P}_n(x)$ and $Q_n(x)$ coincide. Thus the (non-symmetric) polynomials $Q_n(x)$ are obtained from the symmetric polynomials $S_n(x)$ by the Christoffel transform (5.1). This allows expressing the polynomials $Q_n(x)$ in terms of the circle polynomials $\Phi_n(z)$:

$$(5.5) \quad Q_n(x) = z^{-n/2} \frac{\Phi_n^*(z) + z^{1/2} \Phi_n(z)}{1 + z^{1/2}}.$$

The reciprocal transformation from the polynomials $S_n(x)$ to the polynomials $Q_n(x)$ is given by the Geronimus transform

$$(5.6) \quad S_n(x) = Q_n(x) - B_n Q_{n-1}(x),$$

where

$$(5.7) \quad B_n = 2 - A_n = 1 + a_{n-1}.$$

The polynomials $Q_n(x)$ are orthogonal on the interval $[-2, 2]$:

$$(5.8) \quad \int_{-2}^2 Q_n(x) Q_m(x) w(x) (x+2) dx = 0, \quad n \neq m,$$

where the function $w(x)$ is given by (4.7). Formula (5.8) follows from the definition of the Christoffel transform [34].

Consider the special value $Q_n(2)$. From (5.5) we have ($x = 2$ corresponds to $z = 1$)

$$(5.9) \quad Q_n(2) = \Phi_n(1) = (1 - a_0)(1 - a_1) \dots (1 - a_{n-1}).$$

Using these formulas, one can easily obtain an expression for the polynomials $\Phi_n(z)$ in terms of the polynomials $Q_n(x)$:

$$(5.10) \quad \Phi_n(z) = z^{n/2} \frac{(z^{-1/2} - a_n)Q_n(x) - Q_{n+1}(x)}{1 - z^{1/2}}.$$

Note that apart from the polynomials $Q_n(x)$ one can introduce the adjacent polynomials $Q_n^{(-)}(x)$ which are obtained from $S_n(x)$ by the Christoffel transformation which is adjacent with respect to (5.1):

$$(5.11) \quad Q_n^{(-)}(x) = \frac{S_{n+1} - A_n^{(-)}S_n(x)}{x - 2}, \quad A_n^{(-)} = \frac{S_{n+1}(2)}{S_n(2)} = -A_n = 1 - a_{n-1}.$$

The monic polynomials $Q_n^{(-)}(x)$ satisfy the recurrence relation

$$(5.12) \quad Q_{n+1}^{(-)}(x) + b_n^{(-)}Q_n^{(-)}(x) + u_n^{(-1)}Q_{n-1}^{(-)}(x) = xQ_n^{(-)}(x),$$

where

$$(5.13) \quad b_n^{(-)} = a_{n-1} - a_n = -b_n, \quad u_n^{(-1)} = 1 - a_{n-1}^2 = u_n.$$

Comparing (5.12) with (3.8), we see that the polynomials $Q_n^{(-)}(x)$ have the same recurrence coefficients u_n as the polynomials $Q_n(x)$ but with coefficient b_n that have the opposite sign. This means that these polynomials almost coincide:

$$(5.14) \quad Q_n^{(-)}(x) = (-1)^n Q_n(-x).$$

The polynomials $Q_n^{(-)}(x)$ are orthogonal on the interval $[-2, 2]$

$$(5.15) \quad \int_{-2}^2 Q_n^{(-)}(x)Q_m^{(-)}(x)w(x)(x-2)dx = 0, \quad n \neq m.$$

We see that the polynomials $Q_n(x)$ are more convenient to construct the polynomials $\Phi_n(z)$ than the polynomials $S_n(x)$. Indeed, the reflection parameters a_n are immediately determined, up to a sign, from the recurrence coefficient $u_n = 1 - a_{n-1}^2$:

$$(5.16) \quad a_n = \pm \sqrt{1 - u_{n+1}}.$$

Choosing the sign in (5.16) selects the polynomials $Q_n(x)$ or the polynomials $Q_n^{(-)}(x)$.

We can consider a third set of orthogonal polynomials $T_n(x)$ which is obtained from the polynomials $Q_n(x)$ by the Christoffel transform at $x = 2$:

$$(5.17) \quad T_n(x) = \frac{Q_{n+1}(x) - \frac{Q_{n+1}(2)}{Q_n(2)}Q_n(x)}{x - 2} = \frac{Q_{n+1}(x) - (1 - a_n)Q_n(x)}{x - 2}.$$

Simple calculations show that the polynomials $T_n(x)$ are symmetric orthogonal polynomials satisfying the recurrence relation

$$(5.18) \quad T_{n+1}(x) + w_n T_{n-1}(x) = xT_n(x), \quad T_0 = 1, \quad T_1 = x,$$

where

$$(5.19) \quad w_n = (1 + a_{n-1})(1 - a_n).$$

Note that the polynomials $T_n(x)$ can be obtained from the polynomials $S_n(x)$ by a "symmetric Christoffel transform" (see, e.g. [34], [33])

$$(5.20) \quad \begin{aligned} T_n &= \frac{S_{n+2}(x) - \frac{S_{n+2}(2)}{S_n(2)} S_n(x)}{x^2 - 4} = \frac{S_{n+2}(x) - (1 - a_n)(1 - a_{n-1}) S_n(x)}{x^2 - 4} \\ &= \frac{x S_{n+1}(x) - 2(1 - a_{n-1}) S_n(x)}{x^2 - 4}. \end{aligned}$$

Using formula (4.3) and the recurrence relations for the polynomials $\Phi_n(z)$ we can express the polynomials $T_n(x)$ in terms of the polynomials $\Phi_n(z)$:

$$(5.21) \quad T_n(x) = \frac{z^{-n/2}(z\Phi_n(z) - \Phi_n^*(z))}{z - 1}.$$

The polynomials $T_n(x)$ are orthogonal on the interval $[-2, 2]$:

$$(5.22) \quad \int_{-2}^2 T_n(x) T_m(x) (4 - x^2) w(x) dx = 0, \quad n \neq m.$$

Thus with the polynomials $\Phi_n(z)$ orthogonal on the unit circle with real reflection parameters $|a_n| < 2$ one can relate 3 types of polynomials $S_n(x)$, $Q_n(x)$, $T_n(x)$, orthogonal on the interval $[-2, 2]$ of the real line. Two of these polynomials, $S_n(x)$ and $T_n(x)$, are symmetric, whereas the polynomials $Q_n(x)$ are non-symmetric.

Note that the polynomials $S_n(x)$ and $T_n(x)$ coincide with the so-called "companion polynomials" considered in [8]. Formulas (4.4) express the polynomials $\Phi_n(z)$ in terms of the polynomials $S_n(x)$. Note also that the relations between the symmetric polynomials $T_n(x)$ and the non-symmetric polynomials $Q_n(x)$ (without the identification of the connection to OPUCs) were studied by L.M. Chihara and T.S. Chihara in [15].

6. FROM THE JACOBI OPUC TO THE LITTLE -1 JACOBI POLYNOMIALS

The unit circle analogues of the Jacobi polynomials were proposed by Szegő [34] and studied by Golinskii [25] and Badkov [4], [5]. These Jacobi OPUC $\Phi_n(e^{i\theta})$ are orthogonal on the unit circle with respect to the weight function

$$(6.1) \quad \rho(\theta) = (1 - \cos \theta)^{\xi+1/2} (1 + \cos \theta)^{\eta+1/2},$$

where $\theta \in [0, 2\pi]$. It is assumed that $\xi > -1$, $\eta > -1$.

Their parameters a_n have the following expression [4]

$$(6.2) \quad a_n = \begin{cases} \frac{\eta - \xi}{n + \xi + \eta + 2} & \text{if } n \text{ is even} \\ -\frac{1 + \xi + \eta}{n + \xi + \eta + 2} & \text{if } n \text{ is odd} \end{cases}.$$

The condition $a_{-1} = -1$ obviously holds.

Note that the Jacobi OPUC are connected with the generalized Gegenbauer (ultraspherical) polynomials through the DG map.

Indeed, the generalized Gegenbauer polynomials [14], [7] are symmetric orthogonal polynomials satisfying (4.1) with

$$(6.3) \quad v_n = \begin{cases} \frac{n(n+2\xi)}{(n+\xi+\eta)(n+\xi+\eta+1)} & \text{if } n \text{ is even} \\ \frac{(n+2\eta+1)(n+2\xi+2\eta+1)}{(n+\xi+\eta)(n+\xi+\eta+1)} & \text{if } n \text{ is odd} \end{cases},$$

where $\xi > -1$, $\eta > -1$. These polynomials are orthogonal on the interval $[-2, 2]$:

$$(6.4) \quad \int_{-2}^2 S_n(x) S_m(x) (4 - x^2)^\xi |x|^{2\eta+1} dx = 0, \quad n \neq m.$$

Using the Delsarte-Genin map (4.4) and formula (4.2), we can easily verify that the Jacobi OPUC $\Phi_n(e^{i\theta})$ correspond to the generalized Gegenbauer polynomials $S_n(x)$.

Consider now the companion polynomials $Q_n(x)$ corresponding to the parameters (6.2). They are orthogonal on the interval $[-2, 2]$ with respect to the function

$$w(x) = (x + 2)(4 - x^2)^\xi |x|^{2\eta+1}.$$

The recurrence coefficients b_n, u_n for these polynomials are explicitly obtained from formulas (3.9), where the coefficients a_n are given by (6.2).

Choose the parametrization

$$\xi = \frac{\alpha - 1}{2}, \quad \eta = \frac{\beta - 1}{2}.$$

A simple analysis shows that the polynomials $Q_n(x)$ coincide (up to an obvious affine transformation of the argument) with the little -1 Jacobi polynomials $P_n^{(-1)}(x; \alpha, \beta)$ introduced in [37]. The little -1 Jacobi polynomials possess a remarkable property: they are polynomial eigenfunctions of a Dunkl-type differential operator of the first order. Hence these polynomials provide a new nontrivial example of "classical" orthogonal polynomials on the real line (for details see [37]).

Note that similar polynomials were considered in [15], where the recurrence coefficients b_n, u_n were calculated explicitly. These polynomials belong to a special class of orthogonal polynomials and were studied by another approach in [2]. Our approach yields explicit expressions for the recurrence coefficients and the orthogonal polynomials $Q_n(x)$ by a straightforward application of the correspondence formulas between polynomials orthogonal on the unit circle and on the interval.

7. A ONE-PARAMETER GENERALIZATION OF THE COMPANION POLYNOMIALS

Consider a linear pencil of matrices

$$(7.1) \quad K(\lambda) = L + \lambda M$$

with an arbitrary parameter λ .

We have

$$(7.2) \quad K^2(\lambda) = L^2 + \lambda^2 M^2 + \lambda(LM + ML) = 1 + \lambda^2 + \lambda(J^2 - 2) = 1 + \lambda^2 + \lambda H.$$

We see that the Hermitian matrix H can be expressed in terms of $K^2(\lambda)$ for every λ . On the other hand, the matrix $K(\lambda)$ is a nondegenerate symmetric tri-diagonal matrix

$$K(\lambda) = \begin{pmatrix} a_0 + \lambda & r_0 & & & & \\ r_0 & -a_0 + \lambda a_1 & \lambda r_1 & & & \\ & \lambda r_1 & a_2 - \lambda a_1 & r_2 & & \\ & & r_2 & -a_2 + \lambda a_3 & \lambda r_3 & \\ & & & & \ddots & \ddots \end{pmatrix}.$$

Here, nondegenerate means that all off-diagonal entries of the matrix $K(\lambda)$ are nonzero if $\lambda \neq 0$ and $|a_i| \neq 1$.

Hence one can define a family of formal monic orthogonal polynomials $Q_n(x; \lambda)$ depending on the argument x and an additional parameter λ . These polynomials are uniquely defined through the 3-term recurrence relation

$$(7.3) \quad Q_{n+1}(x; \lambda) + b_n(\lambda)Q_n(x; \lambda) + u_n(\lambda)Q_{n-1}(x; \lambda) = xQ_n(x; \lambda),$$

where

$$(7.4) \quad b_n(\lambda) = \begin{cases} a_n - \lambda a_{n-1} & \text{if } n \text{ is even} \\ \lambda a_n - a_{n-1} & \text{if } n \text{ is odd} \end{cases},$$

and

$$(7.5) \quad u_n(\lambda) = \begin{cases} \lambda^2(1 - a_{n-1}^2) & \text{if } n \text{ is even} \\ 1 - a_{n-1}^2 & \text{if } n \text{ is odd} \end{cases}.$$

Note that the eigenvalue problem for the orthogonal polynomials $Q_n(x; \lambda)$ can be presented in algebraic form as

$$(7.6) \quad (L + \lambda M - xI)\vec{q} = 0,$$

where \vec{q} is a vector constructed from the (non-monic) polynomials $Q_n(x; \lambda)$. Equation (7.6) contains two parameters λ and x and belongs to the class of the so-called multi-parameter eigenvalue problems [3], [32].

We already considered the case $\lambda = 1$ that leads to the orthogonal polynomials $Q_n(x)$ with a positive measure on the interval $[-2, 2]$. For general complex values of the parameter λ it is still unclear what the spectral properties of the corresponding orthogonal polynomials are because the matrix $K(\lambda)$ is non-Hermitian. Nevertheless, when λ is real, the polynomials $Q_n(x; \lambda)$ are orthogonal on the real axis with a positive measure. Indeed, as seen from (7.5) the recurrence coefficients u_n are positive for all $n > 0$ and for all real values of λ . In the special case $\lambda = -1$, there is a simple relation between the polynomials $Q_n(x; -1)$ and the polynomials $(-1)^n \Phi_n(-z)$ on the unit circle.

Indeed, in the case $\lambda = -1$, the recurrence coefficients of the polynomials $Q_n(x; -1)$ become

$$(7.7) \quad b_n(-1) = (-1)^n(a_n + a_{n-1}), \quad u_n(-1) = 1 - a_{n-1}^2.$$

The coefficients (7.7) are obtained from the coefficients (5.13) by a simple transformation $a_n \rightarrow (-1)^{n+1}a_n$ corresponding to the monic polynomials $\tilde{\Phi}_n(z) = (-1)^n \Phi_n(-z)$ which are also polynomials orthogonal on the unit circle (since obtained from a simple reflection of the argument z with respect to the real axis). The moments \tilde{c}_n corresponding to the polynomials $\tilde{\Phi}_n(z)$ are $\tilde{c}_n = (-1)^n c_n$.

It is easily verified that

$$A_n \equiv \frac{Q_{n+1}(-2; -1)}{Q_n(-2; -1)} = -(1 + (-1)^n a_n)$$

and hence, one can introduce new symmetric polynomials $\tilde{T}_n(x)$ obtained from $Q_n(x; -1)$ by the Christoffel transform

$$\tilde{T}_n(x) = \frac{Q_{n+1}(x; -1) - A_n Q_n(x; -1)}{x + 2}.$$

These polynomials satisfy the recurrence relation

$$\tilde{T}_{n+1}(x) + \tilde{v}_n \tilde{T}_{n-1}(x) = x \tilde{T}_n(x)$$

with

$$\tilde{v}_n = (1 + (-1)^n a_n)(1 + (-1)^n a_{n-1}).$$

Note that Watkins proposed [35] to consider the unitary linear pencil problem

$$(7.8) \quad (L + \lambda M)\vec{v} = 0$$

which leads to Szegő polynomials orthogonal on the unit circle. In this case $x = 0$ and the parameter λ belongs to the unit circle $|\lambda| = 1$.

8. THE 1-PERIODIC CASE

In order to better understand the nature of the polynomials $Q_n(x; \lambda)$ let us consider the simplest periodic case, that is, $a_0 = a_1 = \dots = 0$ implying that $r_0 = r_1 = \dots = 1$. The corresponding OPUC are simple monomials [31]

$$(8.1) \quad \Phi_n(z) = z^n.$$

The symmetric polynomials $S_n(x)$ obtained by the DG transform (4.3) coincide with the Chebyshev polynomials of the first kind

$$(8.2) \quad S_0 = 1, \quad S_n(x) = 2 \cos(n\theta/2), \quad n = 1, 2, \dots, \quad x = 2 \cos(\theta/2).$$

They are orthogonal on the interval $[-2, 2]$ with the weight function

$$(8.3) \quad w_0(x) = \frac{1}{\sqrt{4 - x^2}}.$$

In order to recover the orthogonality measure for the polynomials $Q_n(x, \lambda)$ we need some additional tools. Namely, let us introduce the following function

$$(8.4) \quad m(z, \lambda) = \langle (z - K(\lambda))^{-1} e_0, e_0 \rangle, \quad z \in \mathbb{C}_+,$$

where $e_0 = (1, 0, \dots)^\top$. It is called the m -function (or Weyl function) of $K(\lambda)$ [31, Section 1.2]. It is a standard fact that

$$(8.5) \quad m(z, \lambda) = -\frac{1}{z - \lambda + m_1(z, \lambda)}, \quad \lambda \in \mathbb{R}_+,$$

where $m_1(z, \lambda)$ is the m -function of the matrix

$$(8.6) \quad K_1(\lambda) = \begin{pmatrix} 0 & \lambda & & \\ \lambda & 0 & 1 & \\ & 1 & 0 & \lambda \\ & & \lambda & \ddots \end{pmatrix}, \quad \lambda \in \mathbb{R}_+.$$

Let us also consider the following family of matrices

$$(8.7) \quad K_{per}(\lambda) = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & \lambda & \\ & \lambda & 0 & 1 \\ & & 1 & \ddots \end{pmatrix} = K(\lambda) - \text{diag}(\lambda, 0, 0, \dots), \quad \lambda \in \mathbb{R}_+.$$

Due to (8.5), we have that

$$(8.8) \quad m(z, \lambda) = \frac{m_{per}(z, \lambda)}{1 + \lambda m_{per}(z, \lambda)}.$$

Next, observe that m_{per} satisfies the following equation

$$(8.9) \quad m_{per}(z) = -\frac{1}{z - \frac{1}{z + \lambda^2 m_{per}(z, \lambda)}},$$

which is equivalent to the algebraic equation

$$(8.10) \quad \lambda^2 z m_{per}^2 + (z^2 - 1 + \lambda^2) m_{per} + z = 0.$$

Therefore, we have that

$$(8.11) \quad m_{per}(z, \lambda) = \frac{1}{2z\lambda^2} \left(-z^2 - \lambda^2 + 1 + \sqrt{(z^2 - \lambda^2 - 1)^2 - 4\lambda^2} \right),$$

where the branch of the square root is chosen such that $m_{per}(z, \lambda) \sim -\frac{1}{z}$ as $z \rightarrow \infty$. Recall that the spectrum $\sigma(K)$ of the operator K is the set of all complex numbers ζ for which $K - \zeta I$ is not invertible. The essential spectrum $\sigma_{ess}(K)$ is defined as $\sigma_{ess}(K) = \sigma(K) \setminus \sigma_d(K)$, where $\sigma_d(K)$ is the set of all isolated eigenvalues with finite multiplicity.

The Floquet theory for semi-finite periodic self-adjoint Jacobi matrices [23] says that the essential spectrum of $K_{per}(\lambda)$ admits the representation

$$(8.12) \quad \begin{aligned} \sigma_{ess}(K_{per}(\lambda)) &= \{z \in \mathbb{R} : (z^2 - (1 + \lambda^2))/\lambda \in [-2, 2]\} = \\ &= [-\lambda - 1, -|\lambda - 1|] \cup [|\lambda - 1|, \lambda + 1]. \end{aligned}$$

Moreover, $K_{per}(\lambda)$ has 0 as an eigenvalue for $\lambda > 1$ [23]. So, according to Weyl's theorem we have that $\sigma_{ess}(K(\lambda)) = \sigma_{ess}(K_{per}(\lambda))$ since $K - K_{per}$ is a self-adjoint one-dimensional operator. Furthermore, $\sigma(K(\lambda)) = \sigma_{ess}(K(\lambda))$ because $m(z, \lambda)$ does not have isolated poles. Thus, from (8.12) we clearly see what the λ -dynamics of the spectrum of $K(\lambda)$ is. Namely, the spectrum starts with two points 1 and -1 at $\lambda = 0$. Then, the points become two different intervals moving towards each other while λ runs from 0 to 1. At the time $\lambda = 1$, the intervals touch each other. After that, they are moving towards infinity in the opposite directions as $\lambda \rightarrow \infty$.

Finally, substituting (8.11) into (8.8) and applying the Stieltjes–Perron inversion formula, we arrive to the fact that the polynomials $Q_n(z, \lambda)$ are orthogonal with respect to the weight function [12]

$$(8.13) \quad w(t) = \begin{cases} \frac{\sqrt{4\lambda^2 - (t^2 - \lambda^2 - 1)^2}}{-\lambda(t^2 - 2\lambda^2 t + \lambda^2 - 1)}, & t \in [|\lambda - 1|, \lambda + 1], \\ \frac{\sqrt{4\lambda^2 - (t^2 - \lambda^2 - 1)^2}}{\lambda(t^2 - 2\lambda^2 t + \lambda^2 - 1)}, & t \in [-\lambda - 1, -|\lambda - 1|]. \end{cases}$$

In particular, for $\lambda = 1$ we get that

$$(8.14) \quad w(t) = \frac{\sqrt{4 - t^2}}{2 - t} = \sqrt{\frac{2 + t}{2 - t}} = w_0(t)(t + 2).$$

From (8.14) it is seen that the polynomials $Q_n(x) = Q_n(x; 1)$ coincide with the Chebyshev polynomials of the third kind [29]. Similarly, for $\lambda = -1$ the polynomials $Q_n(x; -1)$ coincide with the Chebyshev polynomials of the fourth kind.

In fact, the behavior of the set of orthogonality is typical for the entire class of general matrices $K(\lambda)$. Moreover, the weight functions have a similar structure as well. However, before dealing with the general case we have to develop a special technique.

9. FROM THE CHIHARA MAP TO THE GENERIC SDG-MAPS

Here, we recall some well-known results in the theory of orthogonal polynomials. The main goal of this section is to elaborate the machinery for the SDG-maps in the case of arbitrary real λ , i.e. the generic SDG-maps.

Let us start with the following simple lemma

Lemma 9.1. *Let $P_n(x)$ be monic orthogonal polynomials satisfying the recurrence relation*

$$(9.1) \quad P_{n+1}(x) + (\theta - A_n - C_n)P_n(x) + C_n A_{n-1} P_{n-1}(x) = x P_n(x),$$

with the standard initial conditions

$$(9.2) \quad P_0 = 1, \quad P_1(x) = x - \theta + A_0.$$

Assume that the real coefficients A_n, C_n are such that $A_{n-1} > 0$, $C_n > 0$, $n = 1, 2, \dots$ and that $C_0 = 0$. Take θ to be an arbitrary real parameter such that $P_n(\theta) \neq 0$ for $n = 1, 2, \dots$

Define the new polynomials

$$(9.3) \quad \tilde{P}_n(x) = \frac{P_{n+1}(x) - A_n P_n(x)}{x - \theta}.$$

Then the monic polynomials $\tilde{P}_n(x)$ are orthogonal and satisfy the recurrence relation

$$(9.4) \quad \tilde{P}_{n+1}(x) + (\theta - A_n - C_{n+1})\tilde{P}_n(x) + C_n A_n \tilde{P}_{n-1}(x) = x \tilde{P}_n(x).$$

The inverse transformation from the polynomials $\tilde{P}_n(x)$ to the polynomials $P_n(x)$ is given by the formula

$$(9.5) \quad P_n(x) = \tilde{P}_n(x) - C_n \tilde{P}_{n-1}(x).$$

To prove Lemma 9.1 it is sufficient to observe that relation (9.1) provides us with the explicit LU -factorization of the monic Jacobi matrix $J - \theta$ corresponding to the polynomials $P_n(x)$. The rest is an obvious reformulation of the theory of spectral (or Darboux) transformations [42], [10].

Note also that

$$(9.6) \quad A_n = \frac{P_{n+1}(\theta)}{P_n(\theta)}$$

which can easily be verified directly from (9.1).

Remark 9.2. If the polynomials $P_n(x)$ are orthogonal with respect to a weight function $w(x)$ on the interval $[\alpha, \beta]$ then the polynomials $\tilde{P}_n(x)$ are orthogonal with respect to the weight function $(x - \theta)w(x)$ on the interval $[\alpha, \beta]$.

The next Lemma will be useful in the identification of polynomial systems with known families of orthogonal polynomials.

Lemma 9.3 ([13]). *Let $P_n(x)$ and $\tilde{P}_n(x)$ be two systems of orthogonal polynomials defined as in the previous Lemma. Also, assume that the polynomials $P_n(x)$ are orthogonal with respect to a weight function $w(x)$ on the finite interval $[\alpha, \beta]$ and take any $\theta \in (-\infty, \alpha)$. Finally, define the following monic polynomials*

$$(9.7) \quad S_{2n}(x) = P_n(x^2 + \alpha - c^2), \quad S_{2n+1}(x) = (x - \chi)\tilde{P}_n(x^2 + \alpha - c^2),$$

where c is a positive number such that $\sqrt{\alpha - \theta} \leq c$ and χ is a real number defined by the relation $\theta = \chi^2 + \alpha - c^2$. Then the polynomials $S_n(x)$ are orthogonal with respect to the weight function

$$(9.8) \quad \text{sign } x(x + \chi)w(x^2 + \alpha - c^2)$$

on the union of two intervals $[-\sqrt{\beta - \alpha + c^2}, -c] \cup [c, \sqrt{\beta - \alpha + c^2}]$ and satisfy the recurrence relation

$$(9.9) \quad S_{n+1} + (-1)^n \chi S_n(x) + v_n S_{n-1}(x) = x S_n(x),$$

where v_n are given by

$$(9.10) \quad v_{2n} = -C_n, \quad v_{2n+1} = -A_n.$$

The converse statement is also true.

The orthogonality part of this Lemma can be checked by straightforward computations. Then, substituting (9.7) into the corresponding three-term recurrence relations we arrive at (9.9) and (9.10). For more details, see [13].

In what follows, we shall call the polynomials $\tilde{P}_n(x)$, the companion polynomials with respect to $P_n(x)$ (sometimes the polynomials $\tilde{P}_n(x)$ are referred to as the kernel polynomials [14]).

10. A SCHEME FOR REDUCING THE POLYNOMIALS $Q_n(x; \lambda)$ TO λ -INDEPENDENT ORTHOGONAL POLYNOMIALS

In this Section, we show how to reduce the polynomials $Q_n(x; \lambda)$ defined by (7.3) to a λ -independent system of orthogonal polynomials.

First, let us apply Lemma 9.1 to $Q_n(x; \lambda)$ for $\theta = \lambda - 1$ and $\theta = \lambda + 1$. Namely, if we put $\theta = \lambda - 1$ then we can deduce that

$$(10.1) \quad A_n = \begin{cases} -1 - a_n & \text{if } n \text{ is even} \\ -\lambda(1 - a_n) & \text{if } n \text{ is odd} \end{cases}, \quad C_n = \begin{cases} -\lambda(1 + a_{n-1}) & \text{if } n \text{ is even} \\ -1 + a_{n-1} & \text{if } n \text{ is odd} \end{cases}.$$

Next, introducing the polynomials

$$(10.2) \quad \tilde{Q}_n(x; \lambda) = \frac{Q_{n+1}(x; \lambda) - A_n Q_n(x; \lambda)}{x - \lambda + 1}$$

we arrive at the the recurrence relation

$$(10.3) \quad \tilde{Q}_{n+1}(x; \lambda) + \tilde{b}_n \tilde{Q}_n(x; \lambda) + \tilde{u}_n \tilde{Q}_{n-1}(x; \lambda) = x \tilde{Q}_n(x; \lambda),$$

where

$$(10.4) \quad \tilde{b}_n = (-1)^n(\lambda + 1), \quad \tilde{u}_n = -\lambda(1 + a_n)(1 + (-1)^n a_{n-1}).$$

Similarly, for $\theta = \lambda + 1$ we have that

$$(10.5) \quad A_n = \begin{cases} 1 - a_n & \text{if } n \text{ is even} \\ \lambda(1 - a_n) & \text{if } n \text{ is odd} \end{cases}, \quad C_n = \begin{cases} -\lambda(1 + a_{n-1}) & \text{if } n \text{ is even} \\ -1 - a_{n-1} & \text{if } n \text{ is odd} \end{cases}.$$

Then the corresponding transformed polynomials $\tilde{Q}_n(x; \lambda)$ have the recurrence coefficients

$$(10.6) \quad \tilde{b}_n = (-1)^n(\lambda - 1), \quad \tilde{u}_n = \lambda(1 + a_{n-1})(1 - a_n).$$

Clearly, both cases can be rewritten as follows

$$(10.7) \quad \tilde{Q}_{n+1}(x; \lambda) + (-1)^n(d_1 \lambda + d_0) \tilde{Q}_n(x; \lambda) + \lambda u_n^* \tilde{Q}_{n-1}(x; \lambda) = x \tilde{Q}_n(x; \lambda),$$

where d_0 , d_1 , and $u_n^* > 0$ do not depend on λ .

The following Lemma reduces the polynomials $\tilde{Q}_{n+1}(x; \lambda)$ to the polynomials obtained in Lemma 9.3.

Lemma 10.1. *Let the polynomials $\tilde{Q}_n(x; \lambda)$ satisfy (10.7) and be orthogonal with respect to a weight function $w_\lambda(x)$ on the set E_λ for positive λ . Then the monic polynomials defined via*

$$(10.8) \quad S_n(x; \lambda) = (\sqrt{\lambda})^{-n} Q_n(\sqrt{\lambda}x; \lambda)$$

are orthogonal with respect to the weight function $w_\lambda(\sqrt{\lambda}x)$ on the set $E_\lambda^ = \{x : \sqrt{\lambda}x \in E_\lambda\}$ and satisfy*

$$(10.9) \quad S_{n+1}(x; \lambda) + (-1)^n \chi S_n(x; \lambda) + u_n^* S_{n-1}(x; \lambda) = x S_n(x; \lambda),$$

where $\chi = d_1 \sqrt{\lambda} + d_0 \frac{1}{\sqrt{\lambda}}$.

The converse is also true.

The proof is straightforward by making the substitution $x \rightarrow \sqrt{\lambda}x$.

Now, we can apply Lemma 9.3 to represent the polynomials $S_n(x; \lambda)$ in terms of a λ -independent system of orthogonal polynomials. Also, Lemma 9.3 gives us the explicit formula for the weight function of the polynomials $S_n(x; \lambda)$.

Summing up, let us notice that the scheme given in this Section gives us the possibility to reduce the λ -dependent system of polynomials $Q_n(x; \lambda)$ to a system of λ -independent orthogonal polynomials $P_n(x)$ given by (9.1). Namely, we reduced $Q_n(x; \lambda)$ to the polynomials $\tilde{Q}_n(x; \lambda)$ satisfying (10.7). Then Lemma 10.1 leads us to the polynomials S_n , which according to Lemma 9.3 can be constructed from the polynomials $P_n(x)$ given by (9.1).

Finally, it is worth noting that Lemma 9.3 and Lemma 10.1 give an efficient way to construct explicit families of Jacobi matrices of the form

$$J_1 + \lambda J_2,$$

where J_1 and J_2 are bidiagonal matrices. The corresponding eigenvalue problems with spectral parameter x lead to orthogonal polynomials and to Laurent biorthogonal polynomials if we fix x and consider the problem with respect to λ [26], [41].

11. FROM THE JACOBI OPUC TO THE BIG -1 JACOBI POLYNOMIALS

To fit the big -1 Jacobi polynomials in this context, let us consider the polynomials

$$(11.1) \quad Q_n^*(x; \lambda) = g^{-n} Q_n(gx; \lambda).$$

Then (7.3) yields the following

$$(11.2) \quad Q_{n+1}^*(x; \lambda) + b_n^*(\lambda) Q_n^*(x; \lambda) + u_n^*(\lambda) Q_{n-1}^*(x; \lambda) = x Q_n^*(x; \lambda),$$

where the coefficients are as follows

$$(11.3) \quad b_n^* = \frac{1}{g}(\lambda - 1 - A_n - C_n), \quad u_n^* = \frac{1}{g^2} C_n A_{n-1},$$

and A_n , C_n are defined in (10.1).

Next we compute the coefficients A_n and C_n for the reflection parameters a_n given by (6.2). We have

$$(11.4) \quad A_n = \begin{cases} -\frac{n+2+2\eta}{n+2+\xi+\eta} & \text{if } n \text{ is even} \\ -\lambda \frac{n+3+2\xi+2\eta}{n+2+\xi+\eta} & \text{if } n \text{ is odd} \end{cases}, \quad C_n = \begin{cases} -\frac{n+1+2\xi}{n+1+\xi+\eta} & \text{if } n \text{ is even} \\ -\lambda \frac{n}{n+1+\xi+\eta} & \text{if } n \text{ is odd} \end{cases}.$$

Set $\lambda = \frac{1+c}{1-c}$ and $g = -\frac{2}{1-c}$. Now, introducing the new parameters

$$(11.5) \quad A'_n = \frac{1}{g} A_n, \quad C'_n = \frac{1}{g} C_n$$

and choosing the parametrization

$$(11.6) \quad \xi = \frac{\alpha - 1}{2}, \quad \eta = \frac{\beta - 1}{2},$$

we can easily see that the coefficients A'_n and C'_n correspond to those of the big -1 Jacobi polynomials introduced in [38] with a shifted spectral parameter. Recall that the big -1 Jacobi polynomials are polynomials orthogonal on the union of two intervals $[-1, -c] \cup [c, 1]$ with respect to the weight function

$$(11.7) \quad w^{(-1)}(x) = \text{sign } x(x+1)(x-c)(1-x^2)^{(\alpha-1)/2}(x^2-c^2)^{(\beta-1)/2}.$$

Notice that the case $\lambda = 1$ corresponds to $c = 0$. This special case corresponds to the little -1 Jacobi polynomials. In this instance the two intervals of orthogonality connect to a form the single interval $[-1, 1]$.

The big -1 Jacobi polynomials $P_n^{(\alpha, \beta)}(x; c)$ corresponding to the weight function (11.7) are the eigensolutions of [38]

$$(11.8) \quad LP_n^{(\alpha, \beta)}(x; c) = \lambda_n P_n^{(\alpha, \beta)}(x; c),$$

where the operator

$$(11.9) \quad L^{(\alpha, \beta, c)} = g_0(x)(R - I) + g_1(x)\partial_x R$$

with

$$(11.10) \quad g_0(x) = \frac{(\alpha + \beta + 1)x^2 + (c\alpha - \beta)x + c}{x^2}, \quad g_1(x) = \frac{2(x-1)(x+c)}{x},$$

I is the identity operator and R the reflection operator $Rf(x) = f(-x)$.

The eigenvalues of $L^{(\alpha, \beta, c)}$ are

$$(11.11) \quad \lambda_n = \begin{cases} 2n, & n \text{ even} \\ -2(\alpha + \beta + n + 1), & n \text{ odd} \end{cases}.$$

As shown in [39] the operator (11.9) is the most general operator of the first order in ∂_x and involving R that has orthogonal polynomials as eigenfunctions.

We thus see that the orthogonal polynomials $Q_n(x; \lambda)$ obtained by the SDG map from the Jacobi OPUC, satisfy a Dunkl type eigenvalue equation. This equation can easily be obtained from (11.8) by an affine transformation of the argument $x \rightarrow \mu x + \nu$ with appropriately chosen constants μ, ν .

Note that when $\xi = \eta = -1/2$ in (6.2), $a_n = 0, n = 0, 1, 2, \dots$, i.e. this case is equivalent to the 1-periodic case considered in Section 8. This corresponds to the case $\alpha = \beta = 0$ of the big -1 Jacobi polynomials. In particular, the Chebyshev polynomials of the third kind [29] (corresponding to $\lambda = 1$, or equivalently, $c = 0$)

$$V_n(x) = \frac{\cos(n(\tau + 1/2))}{\cos(\tau/2)}, \quad x = \cos \tau$$

coincide with the little -1 Jacobi polynomials [37] with $\alpha = \beta = 0$ and hence they satisfy the Dunkl type eigenvalue equation

$$(11.12) \quad 2(x-1)V'_n(-x) + V_n(-x) = (-1)^n(2n+1)V_n(x).$$

Similarly, for the Chebyshev polynomials of the fourth kind [29]

$$W_n(x) = \frac{\sin(n(\tau + 1/2))}{\sin(\tau/2)}, \quad x = \cos \tau$$

one has the eigenvalue equation

$$(11.13) \quad 2(x+1)W'_n(-x) + W_n(-x) = (-1)^n(2n+1)W_n(x).$$

In fact, equation (11.13) is a simple consequence of (11.12) because of the property $V_n(-x) = (-1)^n W_n(x)$ [29].

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REFERENCES

- [1] M. Anshelevich, *Two-state free Brownian motions*, Journal of Functional Analysis, Vol. 260 (2011), no. 2, 541–565 .
- [2] M. Atia, J. Alaya, A. Ronveaux, *Some generalized Jacobi polynomials*, Comp. Math. Appl. **45** (2003), 843–850.
- [3] F.V. Atkinson, *Multiparameter Eigenvalue Problems*, Academic Press, New York (1972).
- [4] V. M. Badkov, *Uniform asymptotic representations of orthogonal polynomials*, Trudy Mat. Inst. Steklov., Vol. 164 (1983), 6–36; English translation in "Proceedings of the Steklov Institute of Mathematics", **164** (1985), 5–41.
- [5] V. M. Badkov, *Systems of orthogonal polynomials explicitly represented by the Jacobi polynomials*, Math. Notes **42** (1987), 858–863.
- [6] B. Beckermann, M. Derevyagin, and A. Zhedanov, *The linear pencil approach to rational interpolation*, J. Approx. Theory **162** (2010), 1322–1346.
- [7] S. Belmehdi, *Generalized Gegenbauer orthogonal polynomials*, J. Comput. Appl. Math. **133** (2001), 195–205.
- [8] A. Berti, A. Sri Ranga, *Companion orthogonal polynomials: some applications*, Applied Numerical Mathematics **39** (2001) 127–149.
- [9] W. Bryc, W. Matysiak, J. Wesolowski, Quadratic harnesses, q-commutations, and orthogonal martingale polynomials, Trans. Amer. Math. Soc. **359** (2007), no. 11, 5449–5483.
- [10] M.I. Bueno and F. Marcellán, Darboux transformation and perturbation of linear functionals, Lin. Alg. Appl. **384** (2004), 215–242.
- [11] M. J. Cantero, L. Moral, and L. Velázquez, *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle*, Linear Algebra Appl. **362** (2003), 29–56.
- [12] T. S. Chihara, *On co-recursive orthogonal polynomials*, Proc. Amer. Math. Soc. **8** (1957), 899–905.
- [13] T. Chihara, *On kernel polynomials and related systems*, Boll. Unione Mat. Ital., 3 Ser., **19** (1964), 451–459.
- [14] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Mathematics and Its Applications, vol. **13**, Gordon and Breach, New York, 1978.
- [15] L.M. Chihara and T.S. Chihara, *A Class of Nonsymmetric Orthogonal Polynomials*, J. Math. Anal. Appl. **126** (1987), 275–291.
- [16] S. Corteel, R. Stanley, D. Stanton, L. Williams, *Formulae for Askey-Wilson moments and enumeration of staircase tableaux*, arXiv:1007.5174
- [17] M.S. Derevyagin, *The Jacobi matrices approach to Nevanlinna-Pick problems*, J. Approx. Theory, J. Approx. Theory **163** (2011), no. 2, 117–142.
- [18] M.S. Derevyagin, A.S. Zhedanov, *An operator approach to multipoint Padé approximations*, J. Approx. Theory **157** (2009) 70–88.
- [19] P. Delsarte and Y. Genin, *The split Levinson algorithm*, IEEE Trans. Acoust. Speech Signal Process. **34** (1986), 470–478.
- [20] P. Delsarte and Y. Genin, *Tridiagonal approach to the algebraic environment of Toeplitz matrices I. Basic results*, SIAM J. Matrix Anal. Appl. **12**, (1991), 220–238.
- [21] P. Delsarte and Y. Genin, *Tridiagonal approach to the algebraic environment of Toeplitz matrices. II. Zero and eigenvalue problems*, SIAM J. Matrix Anal. Appl. **12**, No. 3 (1991), 432–448.
- [22] F. Essler and V. Rittenberg, *Representations of the quadratic algebra and partially asymmetric diffusion with open boundaries*. J. Phys. **A 29** (1996), 3375–3407. arXiv:cond-mat/9506131.
- [23] J. L. Geronimus *On some difference equations and associated systems of orthogonal polynomials*, Zapiski Math. Otdel. Phys.-Math. Facul. Kharkov Universiteta and Kharkov Mathem. Obchestva, XXV (1957), 87–100 (in Russian).
- [24] G. Gladwell, *Inverse problems in vibration*. Second edition. Solid Mechanics and its Applications, 119. Kluwer Academic Publishers, Dordrecht, 2004.
- [25] B.L. Golinskii, *The V. A. Steklov problem in the theory of orthogonal polynomials*, Mathem. Notes, **15** (1974), 13–19.
- [26] E. Hendriksen and H. van Rossum, *Orthogonal Laurent polynomials*, Indag. Math. (Ser. A) **48** (1986), 17–36.

- [27] S. Kerov, *Interlacing measures*. (English summary) Kirillov's seminar on representation theory, 3583, Amer. Math. Soc. Transl. Ser. 2, 181, Amer. Math. Soc., Providence, RI, 1998.
- [28] R. Killip, I. Nenciu, *Matrix Models for Circular Ensembles*, IMRN, 2004, No. 50, 2665–2701.
- [29] J.C. Mason and D.C. Handscomb, *Chebyshev polynomials*, A CRC Press Company, 2003.
- [30] W. Schoutens, J. L. Teugels, *Lévy processes, polynomials and martingales*, Special issue in honor of Marcel F. Neuts. Comm. Statist. Stochastic Models 14 (1998), no. 1-2, 335–349.
- [31] B. Simon, *Orthogonal Polynomials On The Unit Circle*, AMS, 2005.
- [32] B.D. Sleeman, *Multiparameter spectral theory in Hilbert space*, Pitman, 1978.
- [33] V. Spiridonov and A. Zhedanov, *Discrete-time Volterra chain and classical orthogonal polynomials*, J. Phys A: Math. and Gen., **30** (1997) 8727–8737.
- [34] G. Szegő, *Orthogonal Polynomials*, fourth edition, AMS, 1975.
- [35] D.S. Watkins, *Some perspectives on the eigenvalue problem*, SIAM Review **35** (1993), 430–471.
- [36] A. P. Veselov, A. B. Shabat, *Dressing chains and the spectral theory of the Schrödinger operator*, Funct. Anal. Appl. 27 (1993), No. 281-96.
- [37] L. Vinet and A. Zhedanov, *A “missing” family of classical orthogonal polynomials*, J. Phys. A: Math. Theor. **44** (2011) 085201, arXiv:1011.1669v2.
- [38] L. Vinet and A. Zhedanov, *A limit $q = -1$ for big q -Jacobi polynomials*, Trans. Amer. Math. Soc., to appear, arXiv:1011.1429v3.
- [39] L. Vinet and A. Zhedanov, *A Bochner theorem for Dunkl polynomials*, SIGMA, **7** (2011), 020; arXiv:1011.1457v3.
- [40] A. Zhedanov, *On some classes of polynomials orthogonal on arcs of the unit circle connected with symmetric orthogonal polynomials on an interval*, J. Approx. Theory, **94**, (1998), 73–106.
- [41] A. Zhedanov, *The “classical” Laurent biorthogonal polynomials*, J. Comp. Appl. Math. **98** (1998), 121–147.
- [42] A. Zhedanov, *Rational spectral transformations and orthogonal polynomials*, J. Comput. Appl. Math. **85**, no. 1 (1997), 67–86.
- [43] A. Zhedanov, *Biorthogonal rational functions and the generalized eigenvalue problem*, J. Approx. Theory, **101** (1999), 303–329.

DEPARTMENT OF MATHEMATICS MA 4-5, TECHNISCHE UNIVERSITÄT BERLIN, STRASSE DES 17. JUNI 136, D-10623 BERLIN, GERMANY

CENTRE DE RECHERCHES MATHÉMATIQUES, UNIVERSITÉ DE MONTRÉAL, P.O. BOX 6128, CENTRE-VILLE STATION, MONTRÉAL (QUÉBEC), H3C 3J7, CANADA

DONETSK INSTITUTE FOR PHYSICS AND TECHNOLOGY, 83114 DONETSK, UKRAINE,